Euler Group 5 - Solutions

January 27, 2008

Problem 1

Problem: (*USAMO* 1978/5) Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians can speak at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

Solution: We assume that at most two mathematicians speak a common language. Each mathematician can speak to at most three others, one for each language he knows. Suppose mathematician M_1 can speak with only M_2, M_3, and M_4. Then M_5 can speak with at most three of M_6, M_7, M_8, and M_9. This leaves one of the last four who cannot speak with M_1 or M_5, giving the desired contradiction.

Problem 2

Problem: (*Hungarian Problem Book II*) Let $\overline{AC}$ be the longer of the two diagonals of the parallelogram $ABCD$. Drop perpendiculars from C to $\overline{AB}$ and $\overline{AD}$ extended. If E and F are the feet of these perpendiculars, respectively, prove that

$$AB \cdot AE + AD \cdot AF = (AC)^2$$

Solution: Let G be the foot of the perpendicular to $\overline{AC}$ through B (see figure). Since $\overline{AC}$ divides $ABCD$ into two obtuse triangles, G does not coincide with either A or C, but lies on the segment $\overline{AC}$. The right triangles AEC and AGB are similar because they share $\angle BAC$. Also, right triangles AFC and CGB are similar because $\angle CAD = \angle ACB$. Hence

$$\frac{AC}{AE} = \frac{AB}{AG} \quad \text{and} \quad \frac{AC}{AF} = \frac{BC}{GC},$$

so that

$$AB \cdot AE = AC \cdot AG, \qquad BC \cdot AF = AC \cdot GC,$$

and

$$AB \cdot AE + BC \cdot AF = AC(AG + GC).$$

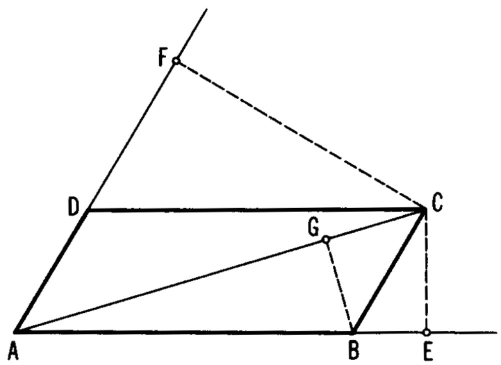
However,

$$BC = AD, \qquad AG + GC = AC,$$

and substitution in the relation above yields

$$AB \cdot AE + AD \cdot AF = (AC)^2$$

as we set out to prove.



This formula implies Pythagoras' theorem as a special case: If $ABCD$ is a rectangle, the formula becomes

$$(AB)^2 + (BC)^2 = (AC)^2.$$

Problem 3

Problem: (*Problem-Solving Strategies* by Engel, referencing R.C. Lyness) Prove that if the difference of the cubes of two consecutive integers can be represented as a square of an integer, then this integer is the sum of the squares of two consecutive integers.

Solution: Multiplying $(x+1)^3 - x^3 = 3x^2 + 3x + 1 = y^2$ by 4, we get

$$3(2x+1)^2 = (2y-1)(2y+1).$$

Since $2y-1$ and $2y+1$ are coprime ($\gcd(m,n) = 1$), we must consider two cases: (a) $2y - 1 = 3m^2, 2y+1 = n^2$, and (b) $2y-1 = m^2, 2y+1 = 3n^2$.

The first case leads to $n^2 - 3m^2 = 2$ which has no solution since it implies $n^2 \equiv -1 \bmod 3$. In the second case, we set $m = 2k+1$ and get

$$2y = 4k^2 + 4k + 2 = 2[(k+1)^2 + k^2],$$

which implies $y = (k+1)^2 + k^2$.