Snoitamrofsnart

Sam Elder

Sunday, February 11, 2007

Sometimes we don't like the diagram that we're given, so we change it in some way. Most of the time, we employ transformations to do this.

There are several types of transformations, and we'll discuss most of them today. These transformations "map" points to other points, often denoted with primes. By superimposing a transformation on the original image, we can prove some interesting things. We will examine the effects of transformations in three coordinate systems: Cartesian coordinates, polar coordinates, and complex numbers. What you get after a transformation is called the **image**. Any points that stay where they are are called **fixed points**. The **trivial transformation** is $z \to z$, or when nothing happens. One major focus will be the aspects of a diagram that are *conserved*, that is, those that don't change whenever we apply a certain transformation. With that, let's start with translations.

1 Translations

Unlike in foreign languages or protein synthesis, a *translation* in math is where you simply move the object somewhere else. Everything is preserved except location. The corresponding points of the new and old figures are all the same distance $\sqrt{a^2+b^2}$ apart, and the corresponding lines are parallel. In Cartesian coordinates, we have $(x, y) \to (x + a, y + b)$.[1] Here's an example:

Problem 1. (AMC 10 2006B) In rectangle $ABCD$, we have $A = (6, -22)$, $B = (2006, 178)$, and $D = (8, y)$, for some integer y. What is the area of rectangle $ABCD$?

Polar coordinates are a lot harder. Basically, the easiest way is to convert into rectangular coordinates, translate, and then convert back. With complex numbers, the conversion is as simple as $z \to z + \lambda$ for $\lambda \in \mathbb{C}$.

Now, for our first hard problem:

Problem 2. (Canada 1997) Let O be a point inside a parallelogram $ABCD$ such that $\angle AOB + \angle COD = \pi$. Prove that $\angle OBC = \angle ODC$.

2 Rotations

While translations are simple in rectangular coordinates, rotations are simple in polar coordinates. First, let's consider rotations about the origin. In polar coordinates, a rotation by angle ϕ about the origin is defined as $r \to r$ and $\theta \to \theta + \phi$. In complex numbers, it is just $z \to ze^{i\phi}$.

Rotations preserve all the properties of the original figure, except when we try to connect the original points and lines to those of the image. In this case, the angle between lines AB and $A'B'$ is always equal to ϕ, the angle of rotation.

[1] This notation tells us that the point (x, y) travels to the point $(x + a, y + b)$. Each translation is defined by the values of a and b.

Problem 3. Prove that a rotation about the origin by angle ϕ corresponds to $(x, y) \to (x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi)$ in rectangular coordinates.

Now let's consider rotation about a point other than the origin. Usually, the easiest thing to do is to translate so that the center of rotation goes to the origin, rotate, and then translate back. For example, a rotation about point A (complex number a) by angle ϕ is equivalent to $z \to (z-a)e^{i\phi}+a$.

A special type of rotation is called a *half-turn.* A half-turn is a rotation by π radians, or $180°$. This rotation is important because the image of a line is parallel to the original. We also know that if $A \to A'$, then the center of rotation is the midpoint of AA'.

Problem 4. (MOP) Given a convex polygon, prove that there exists an equilateral triangle whose vertices lie on the polygon.

Problem 5. (MOP) Let $\triangle ABC$ be an acute triangle, AD an altitude, and $AD = BC$. Let $DEFC$ be a square with A, E, F on the same side of line BC. Show that $BF \perp AC$.

Problem 6. (MOP) Given a point P inside a convex polygon, show that there exist A, B on the polygon such that P is the midpoint of AB.

Problem 7. Triangle ABC is isosceles with $AB = AC$ and has circumcenter O. Points M and N lie on AB and AC respectively such that $BM = AN$. Prove that quadrilateral $ANOM$ is cyclic.

3 Reflections

To *reflect* an object, all you need is a line l. Point A goes to point A' such that the midpoint of AA' lies on l and $AA' \perp l$. The easiest reflections to do in Cartesian coordinates are about the x-axis and the y-axis, which have $(x, y) \to (x, -y)$ and $(x, y) \to (-x, y)$ respectively. Another simple one is across the line $y = x$, which makes $(x, y) \to (y, x)$.

Reflections preserve the sizes, angles, lengths of the original figure. The only lines that are parallel to their images are those parallel or perpendicular to the axis of reflection.

Problem 8. Prove that a reflection about the line $\theta = \phi$ takes $(r, \theta) \to (r, 2\phi - \theta)$.

Problem 9. (WOOT) Line l goes through A and B. Find a general transformation $z \to f(z)$ of the reflection across l (in complex numbers).

Problem 10. Let $\angle XOY$ be an acute angle, and P a point in the interior. Find A on ray OX and B on ray OY such that the perimeter of $\triangle PAB$ is minimal.

4 Dilations

A *dilation* requires a point O and a scalar k. It takes point P to P' on line OP such that $OP' = kOP$. If $k \geq 1$, then P lies on segment OP'; if $0 < k \leq 1$, then P' lies on segment OP; if $k < 0$, then O lies on segment PP'.

A dilation preserves the shape but not the size of any figure. In fact, lengths are scaled by a factor k, areas by a factor k^2, and volumes by a factor k^3. Moreover, each image of a line is parallel to the original.

Problem 11. Find the center of dilation and scale factor that takes $\triangle ABC$ to its medial triangle.

Problem 12. Give the formula for a dilation about point A (complex number a) with scale factor k.

Problem 13. (MOP) Inside a triangle there are circles $\alpha, \beta, \gamma, \delta$ of the same radius. Each of α, β, γ is tangent to two sides of the triangle and to δ (no two are tangent to the same two sides). Show that the incenter and circumcenter of the triangle are collinear with the center of δ.

5 Spiral similarity

A *spiral similarity* is defined as the composition of a rotation and a dilation about the same center. As indicated by the name, any image is similar to the original. The ratios of lengths are preserved, as in a dilation. Here's a hard problem that's actually a lot easier with spiral similarity:
Problem 14. (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

6 Combining transformations

It's easy to see that when a figure is translated twice, the net result is a translation.[2] We can prove this fact by writing each translation in Cartesian coordinates: $(x, y) \to (x + a, y + b)$ and $(x, y) \to (x + c, y + d)$, so the composition is $(x, y) \to (x + a + c, y + b + d)$, which is a translation because it has the same form. With that as an example, let's get to some harder problems.
Problem 15. Prove that the composition of a rotation and a translation is a rotation, provided the rotation is not the identity.
Problem 16. Prove that the composition of two dilations is a dilation.
Problem 17. Prove that the composition of two rotations is either a single rotation or a translation. When does each occur?
Problem 18. Prove that the composition of two spiral similarities is a spiral similarity.

[2]We call such a net result a *composition* of the two transformations.