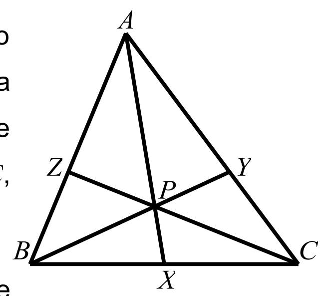
Present complete solutions. No credit will be given for answers without supporting work. Conversely, a minor error that leads to an incorrect answer will not substantially reduce your credit.

Some of these problems may be familiar. If you recognize any of these, and are familiar with their proof, please work on others.

You are only expected to solve one or two problems in the time available during the olympiad prep session. The results of one problem may be used in solving another, regardless of whether you solved the first.

---

1. The line segment joining a vertex of a triangle to any given point on the opposite side is called a *cevian*. Thus, if X, Y and Z are points on the respective sides BC, CA and AB of triangle $\triangle ABC$, the segments AX, BY and CZ are cevians.

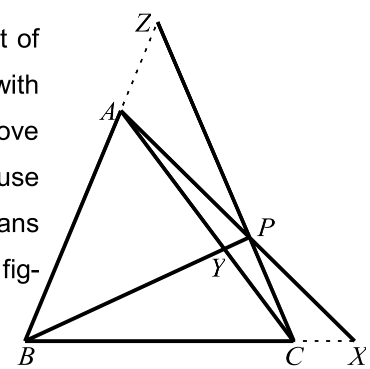


Show that three cevians AX, BY and CZ, one through each vertex of a triangle $\triangle ABC$, are concurrent if and only if:

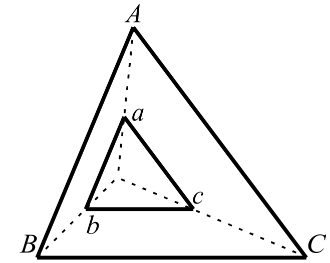
$$\frac{BX}{XC}\,\frac{CY}{YA}\,\frac{AZ}{ZB} = 1$$

where BX, etc. in the formula denotes the length of the corresponding line segment. Three lines are concurrent if they all pass through the same point, labeled P in the diagram.

Bonus: Extend the theorem to allow the point of concurrency P to lie outside triangle $\triangle ABC$, with cevians drawn to extensions of the sides. Prove the extended theorem. The altitudes of an obtuse triangle are one example of extended cevians that are concurrent outside the triangle. The figure shows another example.



2. Let $\triangle ABC$ and $\triangle abc$ be two non-congruent triangles whose sides are respectively parallel, as shown in the figure. Show that the three lines Aa, Bb and Cc are concurrent.



3. For his 100th birthday party George invited 202 guests. They presented him with a rectangular birthday cake with, of course, 100 candles on it. No three candles, nor two candles and a corner of the cake, nor one candle and two corners of the cake were co-linear. George cut the cake into triangular pieces by straight cuts connecting candles with each other and/or with corners of the cake, without any cut crossing previously made cuts. Prove that there are enough pieces to serve each guest a piece of cake, but none will be left for George.

4. Forty-one rooks are placed on a 10x10 chessboard. Prove that you can choose five of them that do not attack each other. (We say that two rooks "attack" each other if they are in the same row or column of the chessboard.)

5. Each of the 49 entries of a square 7x7 table is filled by an integer between 1 and 7, so that each column contains all of the integers 1, 2, 3, 4, 5, 6, 7 and the table is symmetric with respect to its diagonal D going from its upper left corner to its lower right corner. Prove that the diagonal D has all of the integers 1, 2, 3, 4, 5, 6, 7 on it.

6. Is it possible to put 2010 straight line segments on the plane in such a way that each segment has each of its end points lying on inside points of some of the other segments?

Credits:

Problem 1 is, of course, Ceva's theorem. I used *Geometry Revisited* by Coxeter and Greitzer as a reference, but much the same description can be found in any advanced geometry book. The bonus extension comes from my own curiosity. While I've never seen it anywhere, it seems so obvious that it surely must have been published before. Problem 2 is from the same book, exercise 3 of section 1.2.

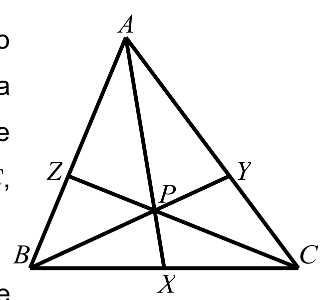
Problems 3 through 6 are all from *Colorado Mathematical Olympiad, The First Ten Years* by Alexander Soifer. Problems 3 and 4 here were problems 4 and 5 at the first CMO, held April 27, 1984. Problems 5 and 6 here were problems 3 and 5 at the second CMO, held April 19, 1985.

Solutions not explicitly credited elsewhere are my own.

Edward A. Gardner, eag at ophidian dot com

1. Ceva's Theorem. Italian mathematician Giovanni Ceva published this in 1678. It's very useful, although not commonly covered in high school Geometry courses. The best way to learn and remember it is to try to prove it. It is another example of something that is relatively straightforward to prove once one realizes there is something worth proving. The term *cevian* of course comes from Ceva.

The line segment joining a vertex of a triangle to any given point on the opposite side is called a *cevian*. Thus, if X, Y and Z are points on the respective sides BC, CA and AB of triangle $\triangle ABC$, the segments AX, BY and CZ are cevians.



Show that three cevians AX, BY and CZ, one through each vertex of a triangle $\triangle ABC$, are concurrent if and only if:

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1$$

where BX, etc. in the formula denotes the length of the corresponding line segment. Three lines are concurrent if they all pass through the same point, labeled P in the diagram.

Solution: From Coxeter and Greitzer, presumably originally from Ceva.

First, show that if the cevians are concurrent, then the formula holds.

Triangles $\triangle ABX$ and $\triangle AXC$ have the same altitude, therefore their areas are proportional to their bases. The same holds for triangles $\triangle PBX$ and $\triangle PXC$. Therefore we have:

$$\frac{BX}{XC} = \frac{ABX}{AXC} = \frac{PBX}{PXC} = \frac{ABX - PBX}{AXC - PXC} = \frac{ABP}{CAP}$$

where BX, etc. denotes the length of the corresponding line segment and ABX, etc. denotes the area of the corresponding triangle.

Similarly:

$$\frac{CY}{YA} = \frac{BCP}{ABP} \qquad \frac{AZ}{ZB} = \frac{CAP}{BCP}$$

Multiplying the three equalities together we get:

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = \frac{ABP}{CAP}\frac{BCP}{ABP}\frac{CAP}{BCP} = 1$$

Now the converse, that if the formula holds then the cevians are concurrent.

Assume that AX and BY meet at P, and that the third cevian through point P is CW. Therefore, by what we just proved we have:

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AW}{WB} = 1$$

But we have assumed that:

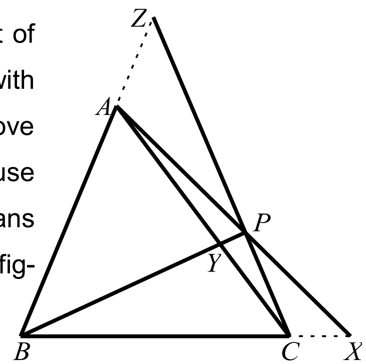
$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1$$

Therefore:

$$\frac{AW}{WB} = \frac{AZ}{ZB}$$

W coincides with Z, and we have proved that AX, BY and CZ are concurrent.

Bonus: Extend the theorem to allow the point of concurrency P to lie outside triangle $\triangle ABC$, with cevians drawn to extensions of the sides. Prove the extended theorem. The altitudes of an obtuse triangle are one example of extended cevians that are concurrent outside the triangle. The figure shows another example.



Solution: The basic approach is the same as before. The challenge is keeping track of details, in particular whether to add or subtract various areas. Standard Geometry has no straightforward way to indicate which side of point A the segment AZ (see figure) lies upon. I chose to introduce the notion of positive and negative lengths. In the figure, BX has positive length, since BX lies on the same side of B as BC, the edge of the triangle. XC has negative length, since XC lies on the opposite side of C as the edge of the triangle. Similarly AZ has negative length while all other segments are positive.

The extension of negative lengths is negative areas. Triangles $\triangle AXC$, $\triangle CZA$, $\triangle PXC$ and $\triangle PZA$ all have negative areas, since their bases XC and AZ have negative lengths.

Triangles $\triangle ABX$ and $\triangle AXC$ have the same altitude, therefore their areas are proportional to their bases. The same holds for triangles $\triangle PBX$ and $\triangle PXC$. Therefore we have:

$$\frac{BX}{XC} = \frac{ABX}{AXC} = \frac{PBX}{PXC} = \frac{ABX - PBX}{AXC - PXC} = \frac{ABP}{CAP}$$

where BX, etc. denotes the length of the corresponding line segment and ABX, etc. denotes the area of the corresponding triangle. Given the figure above, XC, AXC, PXC and CAP are all negative in the equation.

Similarly:

$$\frac{CY}{YA} = \frac{BCP}{ABP} \qquad \frac{AZ}{ZB} = \frac{CAP}{BCP}$$

Given the figure above, AZ and CAP are both negative in the equation.

Multiplying the three equalities together we get:

$$\frac{BX}{XC}\,\frac{CY}{YA}\,\frac{AZ}{ZB} = \frac{ABP}{CAP}\,\frac{BCP}{ABP}\,\frac{CAP}{BCP} = 1$$

I've left the notion of positive and negative lengths mostly to intuition, as a precise definition requires non-standard, confusing notation. In the above, think of BX as meaning "the distance from B to X in the direction of C", which is obviously positive in the figure. Think of XC as meaning "the distance from C to X in the direction of B", which is obviously negative.

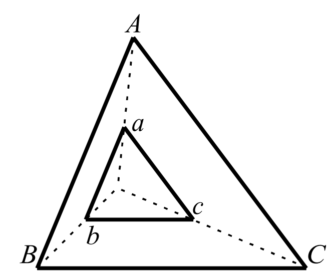
It is straightforward to prove independently that, for an arbitrary point of concurrency P, either:

1. P is inside the triangle, and the cevians from the vertices through P all intersect the unextended sides of the triangle.
2. P is outside the triangle, and exactly two of the extended cevians from the vertices through P intersect an extended side of the triangle. The third cevian intersects an unextended side of the triangle. This is the case shown in the figure.

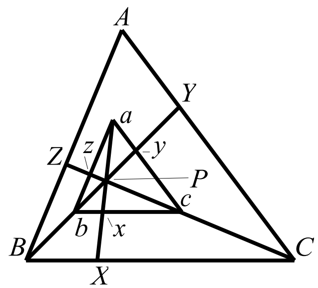
Proof left as an exercise for the reader :-).

2. *Geometry Revisited,* Coxeter and Greitzer, section 1.2, problem 3

Let $\triangle ABC$ and $\triangle abc$ be two non-congruent triangles whose sides are respectively parallel, as shown in the figure. Show that the three lines Aa, Bb and Cc are concurrent.



Solution 1, with Ceva's Theorem: Extend Bb to intersect ac at y and AC at Y. Extend Cc to intersect ab at z and AB at Z. Let P be the intersection of Bb and Cc. Draw aP to intersect bc at x and BC at X.



We have lots of similar triangles:

$\triangle Pbx \sim \triangle PBX$ $\quad \triangle Pxc \sim \triangle PXC$

$$\frac{bx}{BX} = \frac{Px}{PX} \qquad \frac{Px}{PX} = \frac{xc}{XC}$$

$$\frac{bx}{BX} = \frac{xc}{XC} \qquad \frac{bx}{xc} = \frac{BX}{XC}$$

$\triangle bcy \sim \triangle BCY$ $\quad \triangle bya \sim \triangle BYA$

$$\frac{cy}{CY} = \frac{by}{BY} \qquad \frac{by}{BY} = \frac{ya}{YA} \qquad \frac{cy}{CY} = \frac{ya}{YA} \qquad \frac{cy}{ya} = \frac{CY}{YA}$$

$\triangle caz \sim \triangle CAZ$ $\quad \triangle czb \sim \triangle CZB$

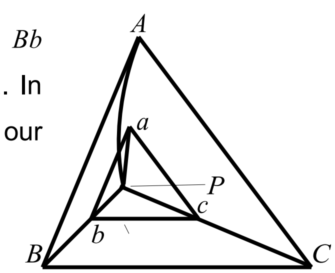
$$\frac{az}{AZ} = \frac{cz}{CZ} \qquad \frac{cz}{CZ} = \frac{zb}{ZB} \qquad \frac{az}{AZ} = \frac{zb}{ZB} \qquad \frac{az}{zb} = \frac{AZ}{ZB}$$

Since Bb, Cc and aP are concurrent at P, from Ceva's Theorem and substituting we have:

$$\frac{bx}{xc}\,\frac{cy}{ya}\,\frac{az}{zb} = \frac{BX}{XC}\,\frac{CY}{YA}\,\frac{AZ}{ZB} = 1$$

which tells us, again from Ceva's Theorem, that line AX also passes through point P. Points a, P and X are co-linear, since we drew aP that way, and we just showed that points A, P and X are co-linear, therefore a, A and P are co-linear and aA passes through point P. QED.

Solution 2, without Ceva's Theorem: Extend Bb and Cc to intersect at point P. Draw Pa and PA. In the figure these are drawn artificially different, our goal is to prove they are the same.



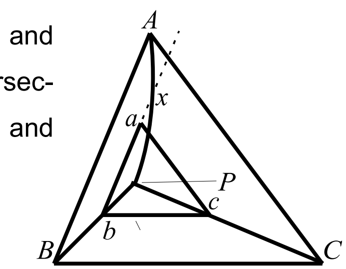
We have several similar triangles:

$\triangle abc \sim \triangle ABC$ $\quad \triangle Pbc \sim \triangle PBC$

$$\frac{ac}{AC} = \frac{bc}{BC} \qquad \frac{bc}{BC} = \frac{Pc}{PC} \qquad \frac{ac}{AC} = \frac{Pc}{PC}$$

We have two sides in the same proportion ($ac : AC$ and $Pc : PC$) and their included angle is the same ($\measuredangle Pca = \measuredangle PCA$), therefore $\triangle Pca \sim \triangle PCA$. This is analogous to using SAS to prove congruence. Therefore $\measuredangle cPa = \measuredangle CPA$, CP and cP coincide, so Pa and PA also coincide. QED.

Solution 3, [Coxeter and Greitzer]: Extend Bb and Cc to intersect at point P. Draw PA, label its intersection with ab point x. Our goal is to prove points a and x are the same.



We have several similar triangles:

$\triangle abc \sim \triangle ABC$ $\quad \triangle Pbc \sim \triangle PBC$ $\quad \triangle Pxb \sim \triangle PAB$

$$\frac{ab}{AB} = \frac{bc}{BC} \qquad \frac{bc}{BC} = \frac{Pb}{PB} \qquad \frac{Pb}{PB} = \frac{xb}{AB} \qquad \frac{ab}{AB} = \frac{xb}{AB}$$

Therefore $ab = xb$ and points a and x coincide, since both lie upon ab the same distance and the same direction from point b. QED.

3. First Colorado Mathematical Olympiad, April 27, 1984, problem 4.

For his 100th birthday party George invited 202 guests. They presented him with a rectangular birthday cake with, of course, 100 candles on it. No three candles, nor two candles and a corner of the cake, nor one candle and two corners of the cake were co-linear. George cut the cake into triangular pieces by straight cuts connecting candles with each other and/or with corners of the cake, without any cut crossing previously made cuts. Prove that there are enough pieces to serve each guest a piece of cake, but none will be left for George.

Solution 1: The corners of each triangular piece correspond to either a candle or a corner of the original cake. Each triangular piece has $180°$ of interior angles filled with cake, so N pieces have $N \cdot 180°$ of interior angles filled with cake. Each candle starts out surrounded by $360°$ of cake and each corner starts out surrounded by $90°$ of cake, so there are $100 \cdot 360 + 4 \cdot 90 = 101 \cdot 360°$ of cake available to fill the interior angles of triangular pieces. Therefore $N \cdot 180 \le 101 \cdot 360$, at most 202 pieces can be cut. Exactly 202 pieces will be cut if George makes cuts to every candle; fewer pieces will be cut if George leaves any candles undisturbed in the middle of a piece. Clearly they should have followed the tradition of including an extra candle for good luck; it was George's bad luck that he didn't get a piece of his own birthday cake.

The first solution in [Soifer] is substantially equivalent to the above.

Solution 2 [Soifer, solution 2]: You are likely familiar with Euler's Formula for polyhedrons, $F + V = E + 2$. Similar formulas apply to other spaces, for the plane $F + V = E + 1$. Here F is the number of triangular pieces and $V = 104$ is the number of candles and corners. $E = \frac{3F + 4}{2}$ is the number of edges; each triangular piece has three edges, but all edges except the four sides of the original rectangular cake are shared by two triangular pieces. Substituting into Euler's Formula we get:

$$F + 104 = \frac{3}{2} F + 2 + 1$$

hence $F = 202$.

4. First Colorado Mathematical Olympiad, April 27, 1984, problem 5.

Forty-one rooks are placed on a 10x10 chessboard. Prove that you can choose five of them that do not attack each other. (We say that two rooks “attack” each other if they are in the same row or column of the chessboard.)

Solution 1: Label the first row of the chessboard with the letters A through J. Label the second row with the same sequence of letters rotated right one column, the third row with the same sequence rotated right two columns, etc. The final result is ten labeled diagonals:

A	B	C	D	E	F	G	H	I	J
J	A	B	C	D	E	F	G	H	I
I	J	A	B	C	D	E	F	G	H
H	I	J	A	B	C	D	E	F	G
G	H	I	J	A	B	C	D	E	F
F	G	H	I	J	A	B	C	D	E
E	F	G	H	I	J	A	B	C	D
D	E	F	G	H	I	J	A	B	C
C	D	E	F	G	H	I	J	A	B
B	C	D	E	F	G	H	I	J	A

There are ten diagonals of ten squares each. With forty-one rooks, at least one diagonal must have five or more rooks (pigeonhole). A rook in a diagonal cannot attack any other rook in the same diagonal. QED.

The second solution in [Soifer], attributed to Russel Shaffer, is substantially equivalent to the above.

Solution 2 [Soifer, solution 1]: With 10 rows and 41 rooks, there must exist a row A containing at least five rooks (pigeonhole). Remove that row, leaving 9 rows and at least 31 rooks.

With 9 rows and at least 31 rooks, there must exist a row B containing at least four rooks (pigeonhole). Remove that row, leaving 8 rows and at least 21 rooks.

Continuing, we get a row C containing at least three rooks (8 rows, 21 rooks), a row D containing at least two rooks (7 rows, 11 rooks) and a row E containing at least one rook (6 rows, 1 rook).

Choose any rook in row E. It must exist since row E has at least one rook. Choose any rook in row D that does not attack the previously chosen rook. One must exist since row D has at least two rooks, and we have only chosen one rook so far. Continue choosing one rook each from rows C, B and A that does not attack the previously chosen rooks. One must exist since each row has at least one more rook than the number previously chosen. QED.

Solution 3 [Soifer, solution 3, attributed to George Berry]: Proof by contradiction. Suppose one can not choose five rooks that do not attack each other, one can only choose four. Choose such a set of four rooks and call them the key rooks.

Exchanging two rows or two columns does not alter the attacks of any rook upon another. If two rooks attack or don't attack each other, they will continue to do so after such an exchange. Use a series of such exchanges to bring the four key rooks into the four top left squares of the main diagonal:

K	?	?	?	*	*	*	*	*	*
?	K	?	?	*	*	*	*	*	*
?	?	K	?	*	*	*	*	*	*
?	?	?	K	*	*	*	*	*	*
*	*	*	*	-	-	-	-	-	-
*	*	*	*	-	-	-	-	-	-
*	*	*	*	-	-	-	-	-	-
*	*	*	*	-	-	-	-	-	-
*	*	*	*	-	-	-	-	-	-
*	*	*	*	-	-	-	-	-	-

K = occupied by a key rook

? = inside square, might or might not be occupied by a rook

* = outside square, might or might not be occupied by a rook

- = cannot be occupied by a rook

The squares marked with a "-" cannot be occupied by a rook. If any were occupied by a rook, that rook plus the four key rooks would be five rooks that do not attack each other, contradicting our assumption.

Consider pairs of diagonally symmetric outside squares marked with a "*". That is, squares $(m\ ,\ n)$ and $(n\ ,\ m)$, where $0 \le m \le 3$ and $4 \le n \le 9$. If both squares of such a pair were occupied by rooks, then those two rooks plus three of the key rooks (the three not attacked by either of the pair) would be five rooks that do not attack each other, which contracts our assumption. Therefore it is not possible for both squares of such a pair to be occupied by a rook.

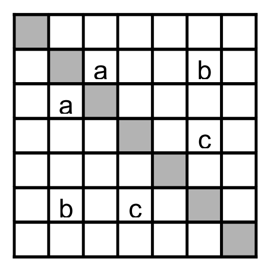
Count the total number of possible rooks. We have 4 key rooks, plus 12 inside squares (marked with "?"), plus one half of the outside squares or 24 rooks. This adds to at most 40 possible rooks. But we have 41 rooks, which is a contradiction. Therefore our assumption that we could choose at most four rooks that do not attack each other is false. QED.

5. Second Colorado Mathematical Olympiad, April 19, 1985, problem 3.

Each of the 49 entries of a square 7x7 table is filled by an integer between 1 and 7, so that each column contains all of the integers 1, 2, 3, 4, 5, 6, 7 and the table is symmetric with respect to its diagonal D going from its upper left corner to its lower right corner. Prove that the diagonal D has all of the integers 1, 2, 3, 4, 5, 6, 7 on it.

Solution: Each column contains the integers 1 through 7, there are seven columns, therefore each of the integers 1 through 7 appears exactly seven times in the table.

Consider pairs of diagonally symmetric squares, such as those shown as a, b or c below. Squares in the main diagonal are shaded.



Since the table is diagonally symmetric, each pair of diagonally symmetric squares contain the same integer. Therefore the table squares not lying on the main diagonal (squares that are not shaded) contain an even number number of each of the integers 1 through 7.

But each integer appears seven times in the entire table, which is odd. Therefore each integer must appear at least once in the main diagonal squares. Since there are seven main diagonal squares and seven integers, each integer must appear exactly once in the main diagonal squares.

The above is substantially equivalent to the solution given in [Soifer].

6. Second Colorado Mathematical Olympiad, April 19, 1985, problem 5.

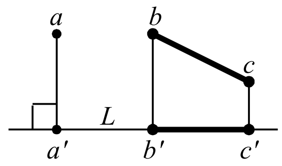
Is it possible to put 2010 straight line segments on the plane in such a way that each segment has each of its end points lying on inside points of some of the other segments?

Solution 1: Consider the convex hull surrounding all the line segments, the convex polygon that would result if an ideally elastic band surrounded the line segments. No part of any line is outside the convex hull. A line may be entirely inside the convex hull, a line may have one or both end points lying on the convex hull, or a line may lie entirely upon the convex hull. If any inside point of a line lies upon the convex hull, then the entire line must lie upon it.

Each vertex of the convex hull is due to an end point of some line segment. That end point cannot lie on an inside point of a line extending outside the convex hull, since no point of any line is outside the convex hull. That end point cannot lie on an inside point of a line lying on the convex hull, since the vertex prevents the entire line from lying upon the hull. Therefore that end point cannot lie on an inside point of any line segment.

Solution 2 [Soifer]: Proof by contradiction. Assume that there is a set S of 2010 straight line segments in the plane such that each end point of each segment is an inside point of some other segment of S.

The projection of a point a on the line L is the base a' of the perpendicular drawn from a to L. The projection of a segment is the union of the projections of all its points (see figure).



Since there are only finitely many line segments in S, we can choose a line L that is not perpendicular to any line segment in S. Let us now project all segments of S onto L and denote the union of all these projections by S'. The first point p' of S' from the right must be the projection of an end point p of one of the line segments in S.

Since the end point p must be an inside point of another line segment, there will be a point q' in S' to the right of p'. This contradiction to p' being the first point of S' from the right proves that such a set of line segments does not exist.