Euler Group 1 - Solutions

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1 Problem 1

Problem: (Engel) Prove that $3^{n+1}|2^{3^n}+1$ for all nonnegative integers n.

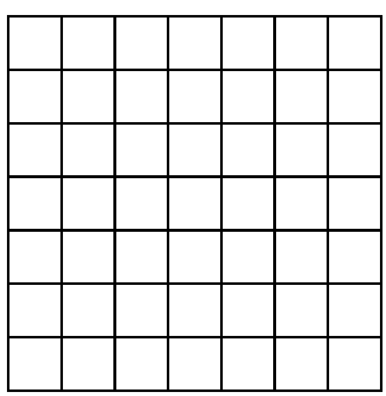
Solution: We induct on n.
Base case. $n=0$. $3^{n+1}=3$, and $2^{3^n}+1=3$, so $3^{n+1}|2^{3^n}+1$ as desired.
Inductive step. Suppose $3^{k+1}|2^{3^k}+1$, that is, let $2^{3^k}+1=m\cdot 3^{k+1}$. We will show that $3^{k+2}|2^{3^{k+1}}+1$. We have

$$\begin{aligned} 2^{3^{k+1}}+1 &= \left(2^{3^k}+1\right)\left(2^{2\cdot 3^k}-2^{3^k}+1\right) && (1)\\ &= \left(2^{3^k}+1\right)\left(\left(2^{3^k}+1\right)^2-3\cdot 2^{3^k}\right) && (2)\\ &= m\cdot 3^{k+1}\left(\left(m\cdot 3^{k+1}\right)^2-3\cdot 2^{3^k}\right) && (3)\\ &= 3^{k+2}\left(m\left(m^2 3^{2k+1}-2^{3^k}\right)\right), && (4)\end{aligned}$$

so $3^{k+2}|2^{3^{k+1}}+1$, as desired.
Induction is complete.

2 Problem 2

Problem: (Engel) A 7×7 square is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?



Solution: Consider the coloring in Figure 1. Each 3×3 tile covers exactly one of the shaded squares.

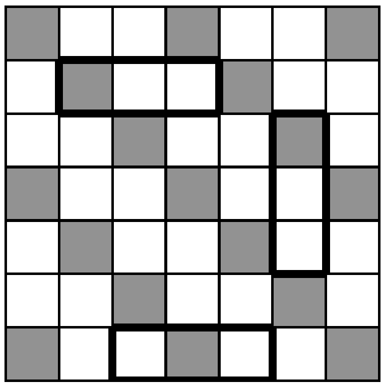


Figure 1

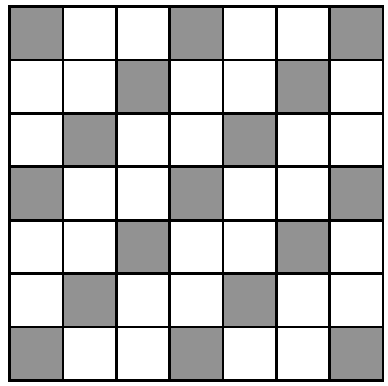


Figure 2

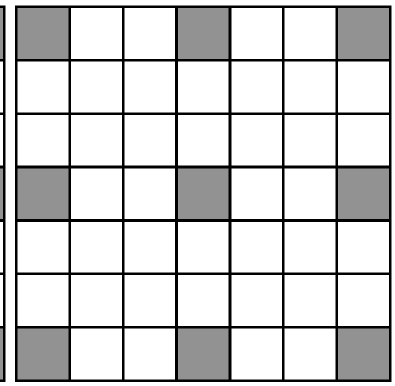


Figure 3

There are 17 shaded squares and only 16 3×1 tiles. Therefore, the 1×1 tile must lie on a shaded square. Now, consider Figure 2. Again, we must have the 1×1 tile on a shaded square. The shaded squares in Figure 3 are covered in both of these colorings. Finally, the following three tilings, and rotations of the first two, show that these squares can in fact be occupied by the 1×1 tile, which is shaded. Therefore the 9 shaded squares in Figure 3 represent the permissible positions of the 1×1 tiles.