

Euler Group 2 - Solutions

October 21, 2007

1 Problem 1

Problem: (USSR Olympiad Problem Book) (a) Prove that if p and $8p - 1$ are both prime, then $8p + 1$ is composite. (b) Prove that if p and $8p^2 + 1$ are both prime, then $8p^2 - 1$ is also prime.

Solution: (a) We need to prove that at least one member of $\{p, 8p + 1, 8p - 1\}$ is composite. Assume p is prime. If $p = 3$, then $8p + 1 = 25$, which is composite, as desired. If $p \neq 3$, then p is congruent to either 1 or 2 modulo 3.

If $p \equiv 1 \pmod{3}$, then $8p + 1 \equiv 0 \pmod{3}$, and $8p + 1 \geq 8 \cdot 7 + 1 = 57$ so $8p + 1$ is composite, as desired.

If $p \equiv 2 \pmod{3}$, then $8p - 1 \equiv 0 \pmod{3}$, and $8p - 1 \geq 8 \cdot 2 - 1 = 15$ so $8p - 1$ is composite, as desired. We are done.

(b) If $p = 3$, then $8p^2 + 1 = 73$ and $8p^2 - 1 = 71$, which are both prime, satisfying the statement.

If $p \neq 3$, then $p \equiv \pm 1 \pmod{3}$. This means that $8p^2 + 1 \equiv 8(\pm 1)^2 + 1 \equiv 0 \pmod{3}$, and since $8p^2 + 1 \geq 33$, it is composite, so the only prime p for which $8p^2 + 1$ is prime is $p = 3$, for which the statement is true.

2 Problem 2

Problem: (Adapted from *The Art and Craft of Problem Solving* by Zeitz) Consider a sequence of eight positive integers containing three distinct integers. Show that there is a consecutive block of integers whose product is a perfect square.

Solution: Let the three distinct numbers be a, b, c , and let the sequence be

$$u_1, u_2, \dots, u_8.$$

We define the successive products

$$p_i = u_1 u_2 \cdots u_i,$$

for $i = 1, 2, \dots, 8$. Each of these eight numbers can be written in the form $a^k b^l c^m$, where k, l, m are nonnegative integers. Consider the parity of the exponents: there are a total of 2^3 different possibilities for the parities of the elements of an ordered triple (k, l, m) . If one of the p_i is of the form (even, even, even), we are done, for p_i would be a perfect square. Otherwise, we are using at most $2^3 - 1$ different parity possibilities for 2^3 different p_i . Hence two of these, say p_i and p_j , where $i < j$, have the same parity values for their exponents. Then the quotient $\frac{p_j}{p_i}$, a product of consecutive integers, will be a perfect square, since all three exponents will be even.

Using the Law of Sines on Olympiad Problems

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1 Introduction

The **Extended Law of Sines** states that for any triangle ABC with side lengths $AB = c$, $BC = a$, and $CA = b$ and circumradius R ,

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C} = 2R. \quad (1)$$

The most common usage of this formula compares two of the sides and their opposite angles, or

$$a \sin \angle B = b \sin \angle A. \quad (2)$$

The Law of Sines can be used to solve for the sides of a triangle given the angles and one of the sides, but that situation is quite rare in olympiad math. Instead, the Law of Sines is most useful for **its ability to relate the lengths and angles of geometric figures**.

2 Examples

2.1 Angle Bisector Theorem

The **Angle Bisector Theorem** states that angle bisector AD of triangle ABC divides the sides of a triangle into the ratio

$$\frac{BD}{CD} = \frac{BA}{CA}.$$

The proof employs the Law of Sines:

By the Law of Sines on triangles ADB and ADC , $\frac{BD}{BA} = \frac{\sin \angle BAD}{\sin \angle BDA}$ and $\frac{CD}{CA} = \frac{\sin \angle CAD}{\sin \angle CDA}$. Since \overline{AD} bisects $\angle BAC$, $\sin \angle BAD = \sin \angle CAD$. Also, angles BDA and CDA are supplementary, so their sines are identical: $\sin \angle BAD = \sin \angle CAD$. As a result,

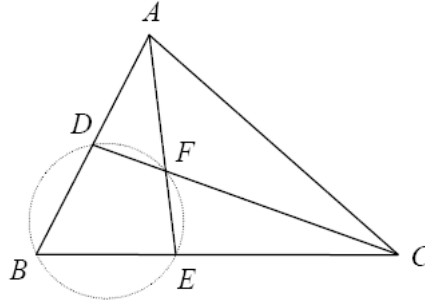
$$\frac{BD}{BA} = \frac{\sin \angle BAD}{\sin \angle BDA} = \frac{\sin \angle CAD}{\sin \angle CDA} = \frac{CD}{CA},$$

the desired result.

2.2 A cyclic quadrilateral proof

I received this problem from Ming Song, who found it on the Hungarian KoMal math competition.

The problem: D is a point of side AB of $\triangle ABC$. E is a point of side BC . AE and CD intersect at F . Prove that if $AE = EC$ and $AB = FC$, then quadrilateral $BDFE$ is cyclic.



My solution: We have

$$\sin \angle DFE = \sin \angle EFC = \frac{EC}{CF} \sin \angle FEC = \frac{AE}{AB} \sin \angle AEB = \sin \angle ABE,$$

so either $\angle DFE = \angle ABC$ or $BDFE$ is cyclic. The first means that $\angle ABC = \angle AFC$, so A, C, F , and B lie on a circle, a contradiction since F lies within $\triangle ABC$. So $BDFE$ is cyclic.

3 Problems

These four problems are selected from Problems in Plane Geometry by Viktor Prasolov, a 495-page geometry problem book.

- 1) Prove that the area S of triangle ABC is equal to $\frac{abc}{4R}$.
- 2) Through point S lines a, b, c and d are drawn; line l intersects them at points A, B, C and D . Prove that the quantity $\frac{AC \cdot BD}{BC \cdot AD}$ does not depend on the choice of line l .
- 3) Quadrilateral $ABCD$ is inscribed into a circle of radius R ; let θ be the angle between the diagonals of $ABCD$. Prove that the area S of $ABCD$ is equal to $2R^2 \cdot \sin A \cdot \sin B \cdot \sin \theta$.
- 4) Triangle ABC is given. On line AB find point M for which the sum of the radii of the circumcircles of triangles ACM and BCM is the least possible one.