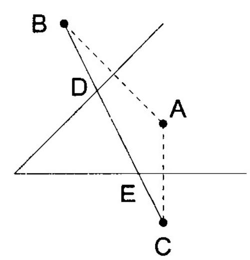
Euler Group 4 Solutions

December 16, 2007

Problem 1: (Fomin) Given 11 different natural numbers, none greater than 20, prove that two of these can be chosen, one of which divides the other.

Solution: We can divide the numbers from 1 through 20 into ten disjoint sets, such that if a pair of numbers is selected from the same set, one of the pair divides the other: $\{11\}$, $\{13\}$, $\{15\}$, $\{17\}$, $\{19\}$, $\{1, 2, 4, 8, 16\}$, $\{3, 6, 12\}$, $\{5, 10, 20\}$, $\{7, 14\}$, $\{9, 18\}$. Then, of any eleven numbers not greater than 20, two of them must fit in one of these pigeonholes, and one of these two divides the other.

Problem 3: (Fomin) Point A, inside an acute angle, is reflected in either side of the angle to obtain points B and C. Line segment BC intersects the sides of the angle at D and E, respectively. Show that $BC/2 > DE$.



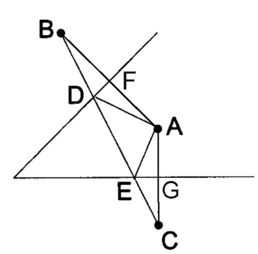
Solution: Draw $\overline{AD}$ and $\overline{AE}$. Label the points where $\overline{AB}$ and $\overline{AC}$ intersect the sides of the angle F and G, respectively. Then $\triangle AFD \cong \triangle BFD$ and $\triangle AGE \cong \triangle CGE$. It follows that BC equals the perimeter of $\triangle ADE$:

$$BC = BD + DE + EC = AD + DE + EA.$$

From the triangle inequality, we know that DE is less than half of the perimeter of $\triangle ADE$:

$$\begin{aligned} DE &< AD + AE \\ 2DE &< AD + AE + DE \\ DE &< \frac{1}{2}(AD + AE + DE) \end{aligned}$$

Therefore $DE < BC/2$.



Euler Group 4 - Solutions

Sam Elder

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Problem 2

Problem: (103 Trigonometry Problems) Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for $k = 1, 2, \ldots$. Prove that

$$f_4(x) - f_6(x) = \frac{1}{12}$$

for all real numbers x.

Solution: We know that

$$\begin{aligned} 1 &= (\sin^2 x + \cos^2 x)^2 \\ &= \sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x \\ &= 4f_4(x) + 2\sin^2 x \cos^2 x \end{aligned}$$

and

$$\begin{aligned} 1 &= (\sin^2 x + \cos^2 x)^3 \\ &= \sin^6 x + 3\sin^4 x \cos^2 x + 3\sin^2 x \cos^4 x + \cos^6 x \\ &= 6f_6(x) + 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x) \\ &= 6f_6(x) + 3\sin^2 x \cos^2 x. \end{aligned}$$

Solving for $f_4(x)$ and $f_6(x)$ and subtracting, we have

$$\begin{aligned} f_4(x) &= \frac{1}{4} - \frac{1}{2}\sin^2 x \cos^2 x. \\ f_6(x) &= \frac{1}{6} - \frac{1}{2}\sin^2 x \cos^2 x. \\ f_4(x) - f_6(x) &= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}, \end{aligned}$$

as desired.

Problem 4

Problem: 2^n and 5^n start with the same digit d. If $n > 0$, what is d?

Solution: Suppose that 2^n has $a + 1$ digits and 5^n has $b + 1$ digits. Because $n > 0$ we have

$$\begin{aligned} d \cdot 10^a < &\quad 2^n &< (d+1) \cdot 10^a \\ d \cdot 10^b < &\quad 5^n &< (d+1) \cdot 10^b \\ d^2 \cdot 10^{a+b} < &\quad 10^n &< (d+1)^2 \cdot 10^{a+b} \\ d^2 < &\quad 10^{n-a-b} &< (d+1)^2 \end{aligned}$$

Since d is a leading digit, $1 \leq d^2 \leq 81$ and $4 \leq (d+1)^2 \leq 100$. Therefore,

$$1 \leq d^2 < 10^{n-a-b} < (d+1)^2 \leq 100$$

and we have $10^{n-a-b} = 10$. This lies between the consecutive squares 9 and 16, so we must have $n = 3$.