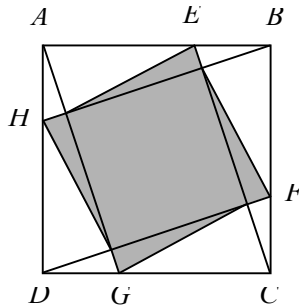


## Mock AIME Two

Ming Song

### Problem 1

$ABCD$  is a square of area 600.  $E$ ,  $F$ ,  $G$ , and  $H$  are the points of  $AB$ ,  $BC$ ,  $CD$ , and  $DA$ , respectively, such that  $AE = 2EB$ ,  $BF = 2FC$ ,  $CG = 2GD$ , and  $DH = 2HA$ . Find the area of the shaded region?



### Problem 4

The sequence  $\{a_n\}$  is defined by  $a_1 = 1$  and for  $n \geq 2$ ,  $a_n = a_{n-1} + 4\sqrt{a_{n-1} + 3} + 4$ . Find  $a_{1000} - 1000 \left\lfloor \frac{a_{1000}}{1000} \right\rfloor$  where  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to  $x$ .

### Problem 5

In a country there are  $n$  ( $n \geq 3$ ) cities. The highways are designed as follows:

- (1) Between any two cities there is one and only one one-way direct highway.
- (2) If there is a direct highway from city  $x$  to city  $y$ , and a direct highway from city  $y$  to city  $z$ , then there is a direct highway from city  $z$  to city  $x$ .

How many cities are there in the country? Justify your answer.

### Problem 7

If  $\sin^2 x^\circ = \sin^2 23^\circ + \cos^2 53^\circ + \sin 23^\circ \cos 53^\circ$ , find the smallest positive value of  $x$ .

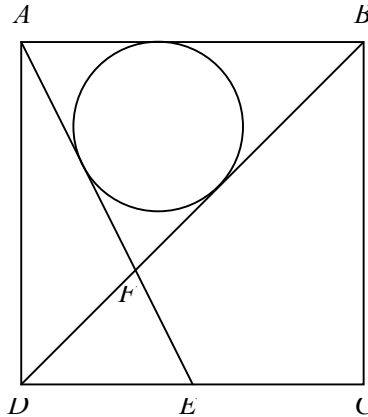
### Problem 8

How many triplets of integers  $(x, y, z)$  are there such that  $1 \leq x < y < z \leq 16$  and  $z - y \leq 5$ ?

*Problem 10*

$ABCD$  is a square of side length 1.  $BD$  is a diagonal.  $E$  is the midpoint of  $CD$ . Draw  $AE$  intersecting  $BD$  at  $F$ . A circle of radius  $r$  is inscribed in  $\triangle ABF$ . If  $r$  can be expressed by

$\frac{m}{\sqrt{n} + \sqrt{k} + s}$  where  $m$ ,  $n$ , and  $k$  are integers, and one of  $n$  and  $k$  is not divisible by any square number, find  $m \cdot n \cdot k \cdot s$ .

*Problem 12*

If  $\gcd(a, b) = 1$  and  $n = \gcd(3a + 4b, 4a - 3b)$  where  $a$ ,  $b$ , and  $n$  are positive integers, find the sum of all possible values of  $n$ .

*Problem 13*

The sequence  $\{a_n\}$  is constructed by the following steps:

Step 1: Write  $a_1 = 1$ : we have 1;

Step 2: Write  $a_2 = 2$ , and then copy the 1 after the 2: we have three terms 1, 2, 1;

Step 3: Write  $a_4 = 3$  following the existing three terms, and then copy all the three terms after the 3: we have seven terms 1, 2, 1, 3, 1, 2, 1;

Step 4: Write  $a_8 = 4$  following the existing seven terms, and then copy all the seven terms after the 4: we have 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1;

etc.

Let  $S_k = \sum_{i=1}^k a_i$  with  $k \geq 1$ . Find  $S_{2009} - 1000 \left\lfloor \frac{S_{2009}}{1000} \right\rfloor$