Catalan Numbers

Colorado Math Circle

April 3, 2010

Balanced Parentheses

Given n pairs of parentheses, in how many ways can you group them so that each open parenthesis has a matching closed parenthesis?

$n = 1$	()	1 way
$n = 2$	()(), (())	2 ways
$n = 3$		
$n = 4$		

Mountain Ranges

How many "mountain ranges" can you form, beginning and ending at "sea level", using n upstrokes and n downstrokes? Here are two examples for $n = 6$:

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    /\/\              /\    /\
 /\/    \/\          /  \  /  \
/          \        /    \/    \
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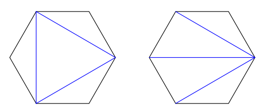
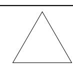
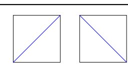
$n=1$	/\	1 range
$n=2$	/\ /\/\, / \	2 ranges
$n=3$		
$n=4$		

Polygon Triangulation

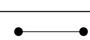
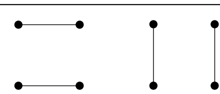
In how many ways can you divide an n-sided polygon into non-overlapping triangles using diagonals? For example, here are two possible triangulations for a hexagon:



$n=3$		1 way
$n=4$		2 ways
$n=5$		
$n=6$		

Hands Across a Circle

If $2n$ people are seated in a circle, in how many ways can they simultaneously shake hands so that each person shakes hands with one other person and none of their arms cross?

$n=1$		1 way
$n=2$		2 ways
$n=3$		
$n=4$		

Dyck Words

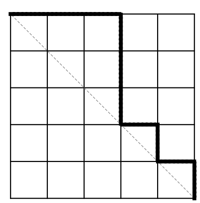
A Dyck word is a string consisting of n X's and n Y's such that, when reading from left to right, there are always at least as many X's as Y's. For example, these are valid Dyck words: XYXXYXYY and XXYXYY. This is not a Dyck word: XXYYYX.

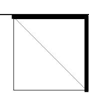
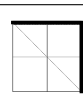
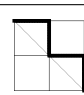
Find the number of Dyck words of length $2n$.

$n = 1$	XY	1 word
$n = 2$	XXYY, XYXY	2 words
$n = 3$		
$n = 4$		

Above-Diagonal Paths

In a grid of $n \times n$ squares, how many paths of length $2n$ lead from the upper left corner to the lower right corner, always staying on or above the main diagonal? Here's one such path for $n = 5$:



$n = 1$		1 path
$n = 2$	 	2 paths
$n = 3$		
$n = 4$		

Multiplication Orderings

Given $n+1$ numbers to multiply together, in how many ways can you order the n multiplications? For example, for $n = 4$, two possible orderings are $a(b(c(d(e))))$ and $a(b(c(d)))(e)$.

$n = 1$	$a(b)$	1 way
$n = 2$	$a(b)(c),\ a(b(c))$	2 ways
$n = 3$		
$n = 4$		

Catalan numbers

Let C_n represent the nth Catalan number. Let $C_0 = 1$.

The first five Catalan numbers are

$$C_0 = 1$$
$$C_1 = 1$$
$$C_2 = 2$$
$$C_3 = \_\_\_\_$$
$$C_4 = \_\_\_\_$$

Can you find a recursive definition for the nth Catalan number?

Can you find a non-recursive formula for the nth Catalan number?

Calculate C_5 and C_6.

References

T. Davis. "Catalan Numbers". `http://www.geometer.org/mathcircles/catalan.pdf`.