

# The Golden Ratio

Colorado Math Circle

September 12, 2009

*Geometry has two great treasures:  
one the Theorem of Pythagoras;  
the other, the division of a line  
into extreme and mean ratio.  
The first we may compare to a measure of gold;  
the second we may name a precious jewel.*

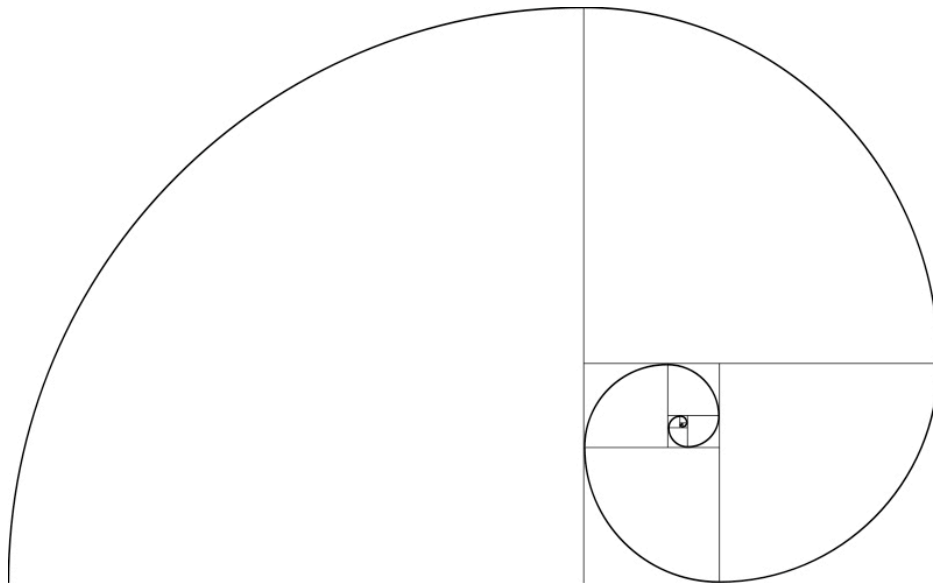
—Johannes Kepler (1571–1630)

*A straight line is said to have been cut  
in extreme and mean ratio when, as the  
whole line is to the greater segment, so is  
the greater to the less.*

—Euclid (300 BC)

*The good, of course, is always beautiful,  
and the beautiful never lacks proportion.*

—Plato (400 BC)



1. Consider the equation

$$\frac{x}{1} = \frac{1}{x-1}$$

- (a) Find the two solutions to this equation. The positive solution is the golden ratio  $\phi$ . Use a calculator to approximate to six decimal places.
- (b) How is the negative solution related to  $\phi$ ?
- (c) What does  $\phi^2$  equal?
- (d) What does  $\frac{1}{\phi}$  equal?

2. The Fibonacci sequence 1, 1, 2, 3, 5, ... is defined recursively as:

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

(a) Find the next 6 Fibonacci numbers.

| $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| 1     | 1     | 2     | 3     | 5     |       |       |       |       |          |          |

(b) Use a calculator to compute the ratio of consecutive Fibonacci numbers to four decimal places:

$$\begin{array}{ccccc} \frac{F_2}{F_1} = & \frac{F_3}{F_2} = & \frac{F_4}{F_3} = & \frac{F_5}{F_4} = & \frac{F_6}{F_5} = \\ \frac{F_7}{F_6} = & \frac{F_8}{F_7} = & \frac{F_9}{F_8} = & \frac{F_{10}}{F_9} = & \frac{F_{11}}{F_{10}} = \end{array}$$

Are the ratios approaching  $\phi$ ?

3. Let

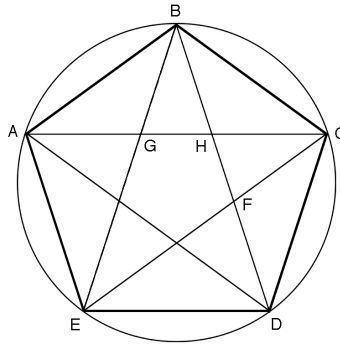
$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$

Show that  $x = \phi$ .

4. Let

$$x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

Show that  $x = \phi$ .



5. In the diagram above, regular pentagon  $ABCDE$  is inscribed in a circle.

(a) Consider  $\triangle EBD$ .

- i. Show that  $\angle EBD = 36^\circ$ .
- ii. Show that  $\angle BED = \angle BDE$ . Find their measure.
- iii. Show that  $\overline{EF}$  is the angle bisector of  $\angle BED$ .
- iv. Show that  $\triangle DEF$  is similar to  $\triangle EBD$ .
- v. Let  $FD = 1$ . Find length  $BF$ .

(b) Consider  $\triangle ABC$ .

- i. Show that  $\angle ABC = 108^\circ$ .
- ii. Show that  $\angle BAC = \angle BCA$ . Find their measure.
- iii. Find  $m\angle ABG$ .
- iv. Show that  $\triangle BGA$  is similar to  $\triangle ABC$ .
- v. Let  $AG = 1$ . Find length  $GC$ .

(c) Compute the following ratios. (*Hint: Make use of symmetry.*)

$$\frac{AC}{GC} = \qquad \frac{GC}{AG} = \qquad \frac{AG}{GH} =$$

6. We can compute the  $n$ th Fibonacci number using Binet's Formula:

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)$$

- (a) Confirm that this formula does indeed compute  $F_1, F_2, F_3, F_4$ . (You may use a calculator.)
- (b) Rewrite Binet's Formula in terms of  $\phi$ .