More Golden Ratio Problems

Whoever cultivates the golden mean avoids both the poverty of a hovel and the envy of a palace.

—Horace (65–8 BC)

1. A *Kepler Triangle* is a right triangle with edge lengths in geometric progression. Let the length of the shorter leg equal 1. Find the lengths of the other two sides.

2. Two different positive numbers a and b each differ from their reciprocals by 1. What is $a+b$?
   (AMC12 2002A)

3. The ratio of the length of a rectangle to its width is the same as that of the diagonal to the length. If the width is 2, how many units are in the length of the diagonal?

4. Equilateral triangle ABC is inscribed in a circle. Points D and E are the midpoints of AB and AC, respectively. Extend $\overline{DE}$ so that it intersects the circle at F. Find the ratio DE/EF.

5. Find b/a if $\log_4 a = \log_6 b = \log_9(a+b)$.

6. Find a if a and b are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.
   (AIME 1988)

7. Given a nonnegative real number x, let $\langle x \rangle$ denote the fractional part of x; that is, $\langle x \rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Suppose that a is positive, $\langle a^{-1} \rangle = \langle a^2 \rangle$, and $2 < a^2 < 3$. Find the value of $a^{12} - 144a^{-1}$.
   (AIME 1997)

8. Let $f(x)$ be a strictly increasing function defined for all $x > 0$ such that $f(x) > -\frac{1}{x}$ and
   $$f(x)\ f(f(x) + 1/x) = 1$$
   for all $x > 0$. Find $f(1)$.
   (USAMTS 2006-07)

9. Prove that
   $$\sin\frac{2\pi}{15} + \sin\frac{4\pi}{15} + \sin\frac{8\pi}{15} + \sin\frac{16\pi}{15} = \frac{\sqrt{15}}{2}$$

10. Find all real numbers x such that
    $$x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$$
    (Canadian Mathematical Olympiad 1998)