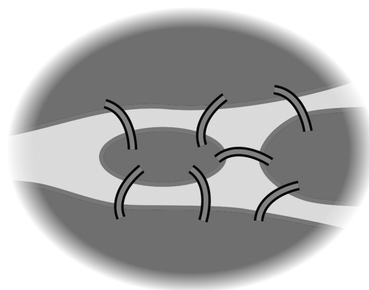
Introduction to Graph Theory

Colorado Math Circle
Galois Group

September 2011



Seven Bridges of Königsberg. The city of Königsberg in Prussia is situated on both sides of the Pregel River, and includes two large islands. The four separate land regions are connected to each other by seven bridges. Is it possible for a resident to walk through the city, cross each of the bridges exactly once, and return home?

Definitions

A **graph** is a collection of points (called **vertices** or **nodes**) and lines (called **edges**) connecting some of them.

When two vertices are endpoints of an edge, we say they are **adjacent**. If the two endpoints of an edge are the same vertex, the edge is called a **loop**. A pair of vertices may have **multiple edges**.

A **simple graph** is a graph containing no loops or multiple edges.

Two graphs are **isomorphic** if they contain the same number of graph vertices and the vertices are connected in the same way.

The **degree** of a vertex is the number of edges touching it.

A **walk** traverses a graph by starting at a vertex and repeatedly moving along an edge to an adjacent vertex. More precisely, a walk is a sequence of vertices $\{v_1, v_2, \ldots, v_n\}$ such that $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ are edges. A walk with no repeated edges is a **trail**.

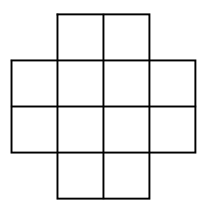
A graph is **connected** if there exists a walk between every pair of vertices.

A **circuit** is a closed trail that begins and ends at the same vertex.

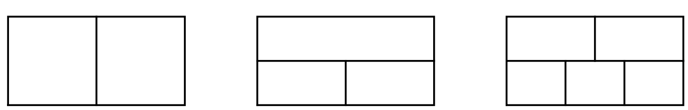
An **Eulerian trail** is a trail that visits each edge in a graph exactly once. An **Eulerian circuit** is a circuit that visits each edge in a graph exactly once.

Exercises

1. *Degree Sum Formula.* Show that the sum of the degrees of the vertices in a graph equals twice the number of edges.
2. *Handshake Lemma.* Show that the number of vertices with odd degree in a graph is even.
3. A computer lab has 9 computers. Can they be connected with cables so that each computer is linked to exactly 3 others?
4. Twenty people attend a party where n handshakes are exchanged. Is it possible for exactly five people to shake an odd number of hands?
5. Fifteen people play in a singles tennis tournament. Show that at the end of the tournament, the number of people who have played an odd number of games is even.
6. Show that if a graph has more than two vertices of odd degree, it is not possible to construct an Eulerian trail.
7. Show that a connected graph has an Eulerian trail if and only if it has either zero or two vertices of odd degree.
8. Show that a connected graph has an Eulerian circuit if and only if all vertices have even degree.
9. *Konigsberg Variations.* What would happen if a new bridge were constructed in Konigsberg? two new bridges?
10. The country of Seven has 15 towns, each of which is connected to at least seven other towns. Prove that one can travel from any town to any other town, possibly passing through some towns in between.
11. Show that every simple graph has two vertices of the same degree.
12. A chessboard is in the form of a cross, obtained from a 4×4 chessboard with corner squares removed. Can a knight travel around the board, land on each square exactly once, and end on the same square it starts from?



13. For each figure below, can you draw a continuous curved line that crosses each line segment exactly once?



References

D. Fomin, S. Genkin, I. Itenberg. *Mathematical Circles (Russian Experience)*. American Mathematical Society, 1996.

"Seven Bridges of Königsberg." *Wikipedia, The Free Encyclopedia.* Wikimedia Foundation, Inc. 24 September 2011.

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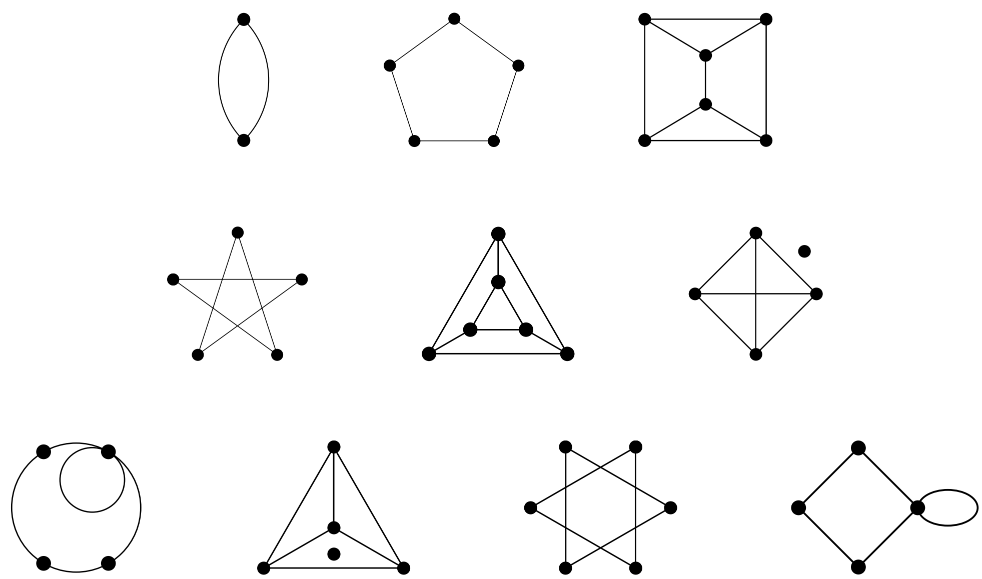
Which of the following graphs are simple? connected? isomorphic?



Do any of these graphs contain an Eulerian trail? an Eulerian circuit?