Knot Polynomials

Colorado Math Circle

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Knot Terminology

A **knot** is a simple knotted loop with no thickness.

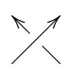
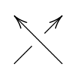
The simplest knot, the unknotted circle $\bigcirc$, is called the **unknot** or **trivial knot**.

A picture of a knot is called a **projection** of the knot.

A **link** is a set of knotted loops joined together.

An **oriented** link includes a direction for traveling around the link.

The **writhe** of an oriented link projection is the number of positive crossings minus the number of negative crossings.

$+1$ crossing:  -1 crossing: 

Calculating the Jones polynomial for a link projection

Let $<\!L\!>$ represent the **bracket polynomial** of link projection L. Let A be the polynomial variable. Then $<\!L\!>$ can be found by applying these three rules:

Rule 1: $< \bigcirc > \; = \; 1$

Rule 2: $< \times > \; = \; A <)(> \; + \; A^{-1} < \asymp >$

$< \times > \; = \; A < \asymp > \; + \; A^{-1} <)(>$

Rule 3: $<\!L \cup \bigcirc\!> \; = \; \left(-A^2 - A^{-2}\right) <\!L\!>$

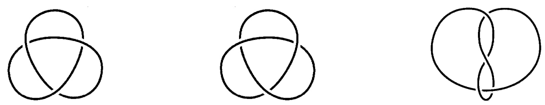
Let $w(L)$ be the writhe of L. Then the ***X* polynomial** (or normalized bracket polynomial) of L is defined to be

$$X(L) = \left(-A^3\right)^{-w(L)} <\!L\!> .$$

The corresponding **Jones polynomial** $V(t)$ can be found by replacing each A in $X(L)$ with $t^{-1/4}$.

PROBLEMS

1. Calculate $< \bigcirc\bigcirc >$.
2. Find the X polynomial and Jones polynomial for the unknot.
3. Show that $X($  $) = X($  $) = X(\bigcirc)$.
4. Calculate the writhe of the knot projections below.



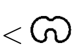
Left trefoil Right trefoil Figure-eight knot

5. Find the Jones polynomial $V(t)$ of the left trefoil knot by executing the following steps.
   (a) Calculate the writhe of the left trefoil knot.
   (b) Compute the bracket polynomial.
   (c) Use the bracket polynomial and writhe to find the X polynomial.
   (d) Use the X polynomial to find the Jones polynomial.
6. What is the Jones polynomial of the right trefoil knot?
7. Find the Jones polynomial of the figure-eight knot.

EXAMPLE

Calculating the bracket polynomial for the Hopf link:

$<$  $>= A <$  $> +A^{-1} <$  $>$
$= A(-A^3) + A^{-1}(-A^{-3})$
$= -A^4 - A^{-4}$

where
$<$  $>= A <$  $> +A^{-1} <$  $>$
$= A(-(A^2 + A^{-2})) + A^{-1}(1) = -A^3.$

$<$  $>= A <$  $> +A^{-1} <$  $>$
$= A(1) + A^{-1}(-(A^2 + A^{-2})) = -A^{-3}.$

REFERENCES

Adams, Colin C. *The Knot Book*. Providence, RI: American Mathematical Society, 2001.