# An Introduction to Knot Theory and Rational Tangles

Colorado Math Circle

December 19, 2009

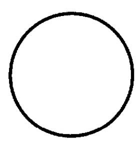
## The Basics

Definition. A **knot** is a simple knotted loop with no thickness.

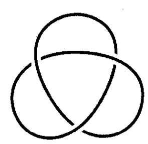
Definition. A picture of a knot is called a **projection** of the knot.

Definition. The places where a knot crosses itself in a projection are called **crossings**.

Definition. A **link** is a set of knotted loops joined together.



The Unknot



Trefoil Knot

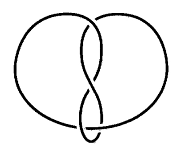
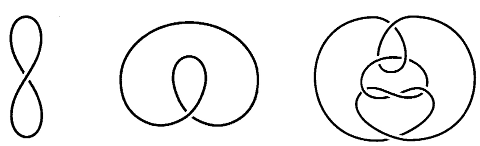
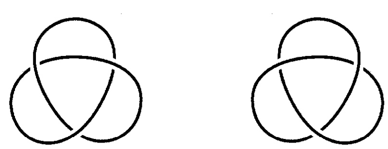


Figure-Eight Knot

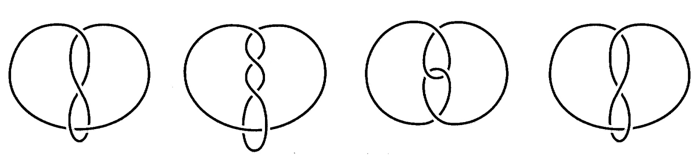
**Exercise 1.** Are the following knots equivalent?



**Exercise 2.** Are these knots equivalent?

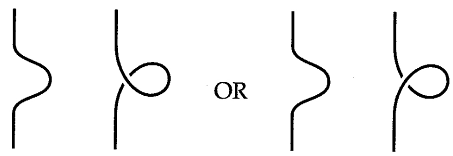


**Exercise 3.** Are these knots equivalent?

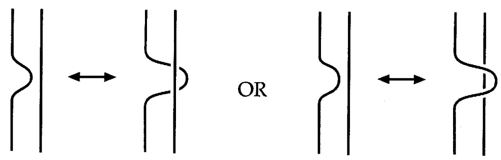


## Reidemeister Moves

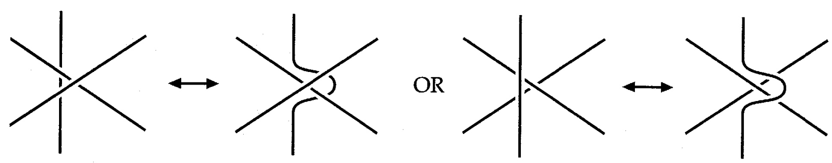
Definition. A **Reidemeister move** is one of three ways to change a projection of a knot that will change the relation between the crossings.




Type I Reidemeister move




Type II Reidemeister move




Type III Reidemeister move

Theorem. A projection of a knot can be changed into another projection of the same knot through a series of Reidemeister moves.

**Exercise 4.** Show that the projections below represent the same knot by finding a series of Reidemeister moves from one to the other.

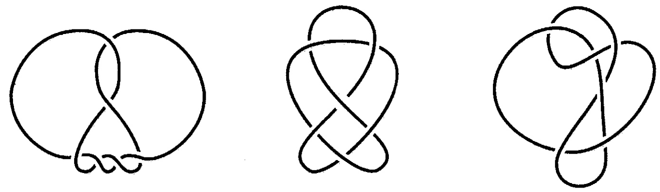
## Tricolorability

Definition. A knot or link is **tricolorable** if each of the strands in the projection can be colored one of three different colors, so that at each crossing, either three different colors come together or all the same color comes together. At least two colors must be used.

Either every projection of a knot is tricolorable or no projection of that knot is tricolorable.

**Exercise 5.** Are the unknot, trefoil knot, and figure-eight knot tricolorable?

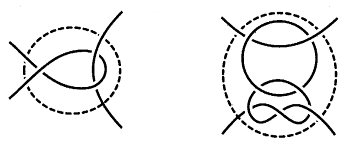
**Exercise 6.** Which of these knots are tricolorable?



$6_1, 6_2, 6_3$ knots

**Exercise 7.** Show that the Reidemeister moves preserve tricolorability.

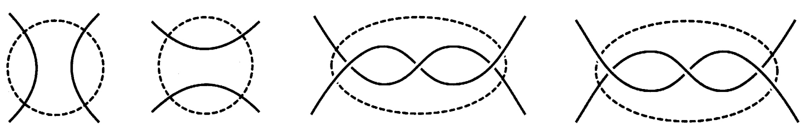
## Conway's Rational Tangles



DEFINITION. A **tangle** is a region of a knot surrounded by a circle such that the knot crosses the circle exactly four times.

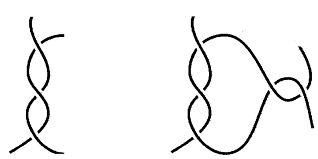
Two tangles are equivalent if we can transform one to the other by a sequence of Reidemeister moves, keeping the four endpoints of the strings fixed.

We assign a positive integer to a tangle if the overstrands have positive slope, and assign a negative integer if the overstrands have negative slope.

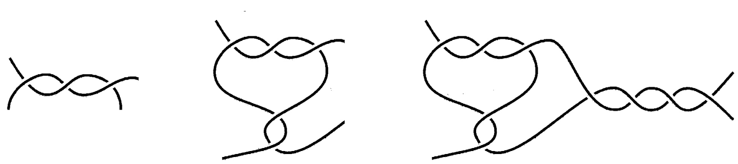


The $\infty$ tangle, 0 tangle, 3 tangle, $-3$ tangle

More complex tangles can be formed by alternately twisting the right-hand (NE-SE) endpoints and bottom (SW-SE) endpoints, always ending with a right-hand twist.



Constructing the $-3\ \ -2$ tangle



Constructing the 3 2 $-4$ tangle

A tangle $a_1\ a_2\ \ldots\ a_n$ corresponds to the **continued fraction**

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\cdots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}.$$

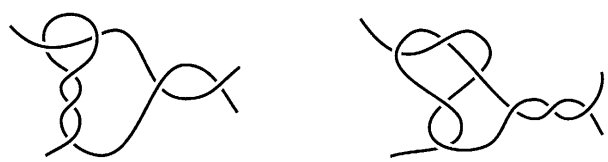
For example, the tangle $-2\ 4\ 3$ corresponds to the continued fraction

$$3 + \cfrac{1}{4 + \cfrac{1}{-2}} = \frac{23}{7}.$$

THEOREM. Two tangles are equivalent if and only if their corresponding continued fractions yield the same rational number.

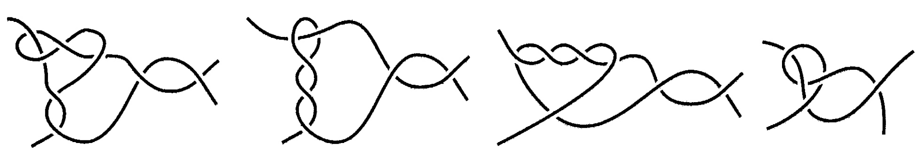
**Exercise 8.** Show that the tangles $2\ 1\ 1$ and $-1\ \ -2\ 2$ are equivalent.

**Exercise 9.** Identify the following tangles. Show that they are equivalent.



**Exercise 10.** Show that the tangles $2\ 1\ a_1\ a_2\ \ldots\ a_n$ and $-2\ 2\ a_1\ a_2\ \ldots\ a_n$ are equivalent.

**Exercise 11.** Which of these tangles are equivalent?



## REFERENCES

Adams, Colin C. *The Knot Book*. Providence, RI: American Mathematical Society, 2001.

Conway, John H. "An Enumeration of Knots and Links." In *Computational Problems in Abstract Algebra* (Ed. J. Leech). Oxford, England: Pergamon Press, pp. 329-358, 1970.

*The KnotPlot Site*. ⟨www.knotplot.com.⟩