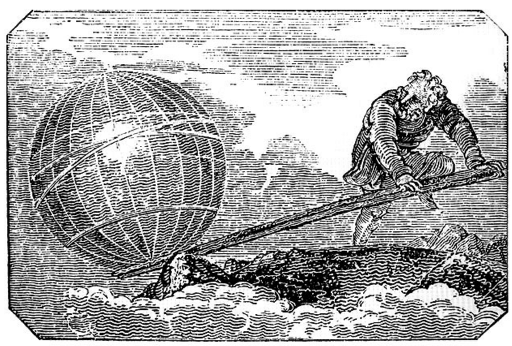
Mass Point Geometry

Colorado Math Circle
Euler Group

September 2011



"Give me a place to stand and I will move the earth." —Archimedes

The underlying idea of the **mass point** technique is the **principle of the lever**. Suppose two masses m_1 and m_2 are placed on a lever. Let d_1 equal the distance from the first mass to the fulcrum, and let d_2 equal the distance from the second mass to the fulcrum. Then the lever will balance if $m_1 d_1 = m_2 d_2$.

Examples

1. (Two cevians) In $\triangle ABC$, points E and D are selected on sides AB and BC, respectively, such that $AE : EB = 4 : 3$ and $CD : DB = 2 : 5$. Segment CE intersects AD at F. Find the ratios $AF : FD$ and $CF : FE$.

2. (Three cevians) In the previous problem, let ray BF intersect AC at G. Find the ratios $AG : GC$ and $BF : FG$.

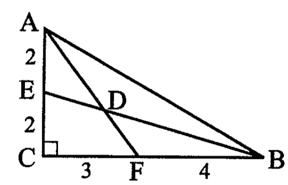
3. (Two cevians) In $\triangle ABC$, points E and D are selected on sides AB and BC, respectively, such that $AE : EB = 2 : 3$ and $CD : DB = 7 : 4$. Segment CE intersects AD at F. Find the ratios $AF : FD$ and $CF : FE$.

4. (Transversal) In $\triangle ABC$, points E, D and G are selected on sides AB, BC and AC, respectively, such that $AE : EB = 4 : 3$, $CD : DB = 5 : 2$, and $AG : GC = 2 : 1$. If BG intersects ED at F, find the ratios $EF : FD$ and $BF : FG$.

5. (Transversal) In $\triangle ABC$, let E be on AB such that $AE : EB = 1 : 3$, D on BC such that $BD : DC = 2 : 5$, and F on ED such that $EF : FD = 3 : 4$. Finally, let $\overrightarrow{BF}$ intersect AC at G. Find $AG : GC$ and $BF : FG$.

6. (Angle bisector) In $\triangle ABC$, points D and E are on AB and AC, respectively. The angle bisector of $\angle A$ intersects DE at F and BC at T. If $AD = 1, DB = 3, AE = 2$, and $EC = 4$, compute the ratio $AF : AT$. (1992 ARML Individual #8)

7. (Area) In the figure, what is the area of $\triangle ABD$? Express your answer as a common fraction. (2008 National MATHCOUNTS Team #10)



8. (Two center points) In $\triangle ABC$, points D and E are on sides BC and CA, respectively, and points F and G are on side AB such that F is between A and G. BE intersects CF at point O_1 and BE intersects DG at point O_2. If $FG = 1, AE = AF = DB = DC = 2$, and $BG = CE = 3$, compute $\dfrac{O_1O_2}{BE}$.

9. (Trisection points) In $\triangle ABC$, points D, E, and F are the trisection points of AB, BC, and CA nearer to A, B, and C, respectively.

   (a) If BF intersects AE at point J, find $BJ : JF$ and $AJ : JE$.

   (b) If CD intersects AE at point K and intersects BF at point L, show that $DK : KL : LC = EJ : JK : KA = FL : LJ : JB = 1 : 3 : 3$.

   (c) Show that the area of $\triangle JKL$ is 1/7 the area of $\triangle ABC$.

10. (13-14-15 triangle) The sides of $\triangle ABC$ are $AB = 13, BC = 15$, and $AC = 14$. Let BD be an altitude of the triangle. The angle bisector of $\angle C$ intersects side AB at F and altitude BD at E. Find $CE : EF$ and $BE : ED$.

Problems

1. Points E and D are selected on sides AB and BC of $\triangle ABC$ such that $AE : EB = 1 : 3$ and $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Find $\dfrac{EF}{FC} + \dfrac{AF}{FD}$. (1965 ASHME #37)

2. In $\triangle ABC$, C' is on side AB such that $AC' : C'B = 1 : 2$, and B' is on AC such that $AB' : B'C = 3 : 4$. If BB' and CC' intersect at P, and if A' is the intersection of ray AP and BC then find $AP : PA'$. (1975 NYCML #27).

3. In $\triangle ABC$, point D is on segment AC with $AD : DC = 2 : 1$, and point E is on segment AB with $AE : EB = 2 : 3$. Segments EC and DB intersect at point K. What is $DK : KB$? Express your answer as a common fraction. (2011 National MATHCOUNTS Team # 9)

4. In $\triangle ABC$, D is on AB such that $AD : DB = 3 : 2$ and E is on BC such that $BE : EC = 3 : 2$. If ray DE and ray AC intersect at F, then find $DE : EF$. (1976 NYCML #13)

5. In $\triangle ABC$, angle bisectors AD and BE intersect at P. If the sides of the triangle are $a = 3, b = 5, c = 7$, with $BP = x$, and $PE = y$, compute the ratio $x : y$. (1989 ARML Team #4)

6. A circle is inscribed in a 3-4-5 triangle. A segment is drawn from the smaller acute angle to the point of tangency on the opposite side. This segment is divided in the ratio $p : q$ by the segment drawn from the larger acute angle to the point of tangency on its opposite side. If $p > q$ then find $p : q$. (1977 NYCML)

7. In $\triangle ABC$, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB, respectively, and lines EF and AM intersect at G. If $AE = 2AF$ then find EG/GF. (1975 AHSME #28)

8. In $\triangle ABC$, $\angle CBA = 72^\circ$, E is the midpoint of side AC and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. Find the ratio of the area of $\triangle BDF$ to the area of quadrilateral $FDCE$. (1980 AHSME #21)

9. In $\triangle ABC$, cevians AD, BE and CF intersect at point P. The areas of $\triangle PAF$, $\triangle PFB$, $\triangle PBD$, and $\triangle PCE$ are 40, 30, 35, and 84, respectively. Find the area of $\triangle ABC$. (1985 AIME #6)

10. In $\triangle ABC$, the bisector of $\angle B$ intersects AC at D and intersects median AM at E. If $\sin A = 4/5$ and $\sin C = 24/25$, find $AE : EM$ and $BE : ED$.

11. The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If AD is the altitude to the side of length 14, what is the ratio $HD : HA$? (1964 AHSME #35)

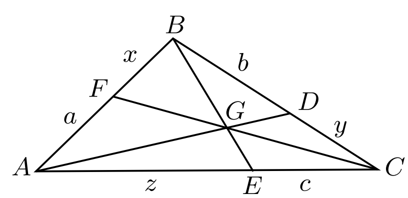
Extras

1. (Centroid) Show that the medians of a triangle are concurrent in a point that divides each median in a ratio of 2:1.

2. (Varignon's Theorem) Show that the four midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

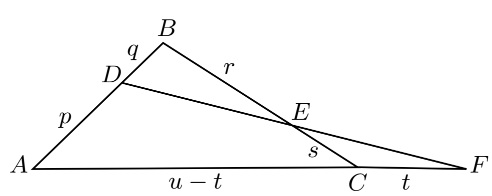
3. (Incenter) Show that the center of mass of a triangle is located on each angle bisector, and that the three angle bisectors are concurrent at this point, called the *incenter*.

4. (Ceva's Theorem) Show that the three cevians AD, BE, and CF in $\triangle ABC$ are concurrent if and only if the product of the three ratios $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, or $abc = xyz$. (See figure below.)



5. (Menelaus' Theorem) If a transversal is drawn across three sides of a triangle (extended if necessary), the products of the non-adjacent segments are equal. In the figure below,

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1 \quad \text{or} \quad AD \cdot BE \cdot CF = DB \cdot EC \cdot FA.$$



From *A Decade of the Berkeley Math Circle* by Zvezdelina Stankova and Tom Rike, Volume 1, MSRI-AMS, 2008.