

COLORADO MATH CIRCLE: TOPICS IN NUMBER THEORY

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- (1) Let L and W be positive integers. Suppose we have a rectangle of length L and width W , made of a reflective material. If we shoot a laser into the rectangle from one corner at a 45 degree angle, how many times will the beam be reflected before hitting another corner? What if L and W are positive real numbers instead? What if we change the angle?
- (2) Let n and k be positive integers. Draw n vertices in a circular arrangement and draw an edge between any two vertices which are k steps apart. We will call this graph $A(n, k)$.

Recall that two vertices of a graph are said to be in the same *component* if there is a connected sequence of edges leading from one vertex to the other.

- (a) How many components does $A(n, k)$ have? In other words, what is the minimum number of times that you have to pick up your pencil while drawing the edges of $A(n, k)$? When does $A(n, k)$ have only one component?
 - (b) We could also make a *directed* version of $A(n, k)$ by drawing an *arrow* (instead of an edge) from each vertex v to the vertex which is k steps clockwise from v . How many directed $A(n, k)$'s are there with d components?
- (3) Suppose you are standing at the origin of the Euclidean plane with tall pillars located at all the non-zero integer lattice points.
 - (a) What do you see?

For example, you can see the pillar at (1,1), but it blocks all the pillars directly behind it e.g. (2, 2), (3, 3) ...

- (b)(★) What percent of the pillars are visible?
 - (c)(★) What about the same questions for the grid made up of equilateral triangles? How about three-dimensional integer lattice points?
- (4) Let n and k be positive integers. Draw n vertices in a circular arrangement, label them 0 to $n - 1$ (reading clockwise, let's say), and draw an edge from vertex k to the vertex which is $2k$ steps clockwise from 0. We will call this graph $M(n, 2)$.

Open Problem:(★★) How many components does $M(n, 2)$ have?

- (5) (★★) Let n be an integer. When can n be expressed a sum of two square integers?