Olympiad Problems and Solutions

Charles Xu

chxu616@gmail.com

Inequalities

Solving olympiad-level inequality problems starts with knowing which of a number of favorite inequalities to apply. These are the most useful:

- **Jensen's inequality** likely finds the broadest application, albeit often in the specific form of AM-GM or weighted AM-GM (see below). Jensen's states that if a function f is convex on an interval I (i.e., if $x_1, x_2 \in I$ then the segment from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is contained in the region $\{(x, y) : y \geq f(x)\}$, as for $f(x) = x^2$), then for $x_1, x_2, ..., x_n \in I$,

  $$f(\frac{x_1 + x_2 + ...x_n}{n}) \leq \frac{1}{n}(f(x_1) + ... + f(x_n))$$

  or, with a more general weighted average,

  $$f(\sum_{i=1}^{n} \omega_i x_i) \leq \sum_{i=1}^{n} \omega_i f(x_i)$$

  where the ω_i add to 1. Unless $f(x)$ is linear, equality occurs only with all x_i equal. The case with $n = 2$ follows self-evidently from the definition of a convex function, and the general inequality is easily proved by induction on n.

- The **AM-GM inequality**,

  $$\frac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

  and its weighted form (with the ω_i summing to 1)

  $$\sum_{i=1}^{n} \omega_i x_i \geq \prod_{i=1}^{n} x_i^{\omega_i}$$

  both follow easily from Jensen's inequality applied to the logarithms of the x_i. Again, we only have equality with the x_i equal. This can be generalized to the

- **Power mean inequality**, which states that

  $$(\sum_{i=1}^{n} x_i^p)^{1/p} \geq (\sum_{i=1}^{n} x_i^q)^{1/q}$$

  iff $p \geq q$, with equality only for all x_i equal as usual. (The generalized power mean approaches the geometric mean in the limit as the exponent p approaches 0.) $p = 1, q = 0$ gives AM-GM; $p = 2, q = 1$ gives the RMS-AM inequality and $p = 0, q = -1$ gives the GM-HM inequality.

- The **Cauchy-Schwarz inequality** states that

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1b_1 + a_2b_2 + ... + a_nb_n)^2$$

with equality iff all the ratios $\frac{a_i}{b_i}$ are equal. This can be easily proved from the trivial inequality $(a_ib_j - a_jb_i)^2 \geq 0$ for $i \neq j$ or by treating $\{a_1, ...a_n\}$ and $\{b_1, ..., b_n\}$ as vectors.

- The **rearrangement inequality** states that if $x_1 \leq x_2 \leq ... \leq x_n$ and $y_1 \leq y_2 \leq ... \leq y_n$, then

$$x_1y_1 + x_2y_2 + ...x_ny_n \geq x_{i_1}y_{j_1} + ... + x_{i_n}y_{j_n} \geq x_1y_n + x_2y_{n-1} + ... + x_ny_1$$

where $\{i_1, ..., i_n\}$ and $\{j_1, ...j_n\}$ are arbitrary permutations of $\{1, 2, ..., n\}$.

- **Smoothing** is a useful technique when you're given that the variables in an inequality sum to a constant value. For example, if replacing x and y with $x - \delta$ and $y + \delta$ respectively (with $x > y$ and $\delta > 0$) strictly increases the value of one side of the inequality, you can argue that that side reaches its maximum value if all the variables are set equal to their arithmetic mean. You can also smooth two variables apart (replacing x and y with $x + \delta$ and $y - \delta$) to find the opposite extremum.

- Likewise, **compactness** is a useful argument when each variable is limited to a finite interval. If one side of the inequality is convex when considered as a function of each variable (holding the others constant), then it reaches a maximum when each variable is at an endpoint of its respective interval.

Now take a crack at these problems. Recall the conditions for smoothing and compactness in particular.

1. (IMO 1984) For $x, y, z > 0$ and $x + y + z = 1$, prove that $xy + yz + xz - 2xyz \leq \frac{7}{27}$.

2. (Bulgaria 1995) Let $n \geq 2$ and $0 \leq x_i \leq 1$ for i from 1 to n inclusive. Show that

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_nx_1) \leq \lfloor \frac{n}{2} \rfloor$$

and determine when there is equality.

3. (MOP 2008) Let a, b, c be positive real numbers with $a + b + c \geq 1$. Show that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{1}{2}.$$

4. (MOP 2008) Let $a_1, a_2, ..., a_n \geq 0$ be real numbers summing to 1. Prove that $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n \leq \frac{1}{4}$.

5. Challenge (USAMO 1998): The reals $x_1, x_2, ..., x_{n+1}$ satisfy $0 < x_i < \frac{\pi}{2}$ and $\sum_{i=1}^{n+1} \tan(x_i - \frac{\pi}{4}) \geq n - 1$. Show that $\prod_{i=1}^{n+1} \tan x_i \geq n^{n+1}$.

Geometry

Olympiad geometry can draw upon a large number of results, most of which you should already have encountered. Particularly common are concurrence problems requiring the use of (usually) Ceva's Theorem and problems about the cyclicity of quadrilaterals, for which the angle conditions (opposite angles add to 180°, angles intercepting a common side are equal) are critical.

You may not be as accustomed to the use of geometric transformations in proofs (I certainly wasn't going into MOP), so be sure to understand the principle behind these most common ones:

- You should already be familiar with the isometric (length- and angle-preserving) transformations of **translation** along a vector, **reflection** across a line, and **rotation** about a point through an angle.

- **Homethety/dilation** about a point P with scale factor r (which can be negative) takes a point Q to its image Q' on line PQ such that $PQ' = r \cdot PQ$, where these lengths are directed so as to allow for negative scale factor. This preserves angles, parallel lines, and length ratios.

- **Spiral similarity** about a point P with scale factor r and angle θ composes a dilation and a rotation.

- **Inversion** about a point P with radius r takes a point Q to Q' on line PQ such that $PQ \cdot PQ' = r^2$. This takes circles not through P to other circles not through P, circles through P to lines not through P and vice versa, and leaves lines through P unchanged.

Using these tranformations in the following problems may not be immediately intuitive, but nonetheless reduces their complexity considerably:

1. (Euler line) Prove that the centroid G of a triangle lies between the orthocenter H and the circumcenter O such that $OG = 2HG$.

2. (Fermat point) $\triangle ABC$ has no angle greater than $120°$, and F is the point in the interior such that $PA + PB + PC$ is minimum. Prove that $\angle APB = \angle BPC = \angle CPA = 120°$.

A surprising number of problems can also be solved by brute-force angle-chasing or area-chasing, as below:

3. (MOP 2008 homework) Let $ABCD$ be a convex quadrilateral with longest side AB. Points M and N are located on sides AB and BC, respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

Number Theory

Most olympiad number theory problems are solved by direct if involved application of these theorems:

- **Bezout's Theorem:** Let the greatest common divisor (a, b) of a and b be d. Then the equation $ax + by = n$ has a solution in integers x and y iff $d|n$. In particular, the lowest positive value of $ax + by$ is d.

- **Euclidean Algorithm:** Let $a \geq b$ and $d = (a, b)$. Let r be the remainder upon dividing a by b, so that $a = qb + r$. If $r \neq 0$, replace a and b with b and r respectively and find the new remainder. Repeat until $r = 0$ and we have $m = pn$. Then $n = d$.

- **Chinese Remainder Theorem:** Let $m_1, m_2, ..., m_n$ be pairwise relatively prime natural numbers. Then for any n integers $a_1, a_2, ..., a_n$ the n congruences

$$x \equiv a_i \bmod m_i$$

  have a common solution x, and any two common solutions are congruent mod $m_1 m_2 ... m_n$.

- **Euler's Theorem and Fermat's Little Theorem:** If $(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \bmod n$, where $\phi(n)$ is the number of positive integers less than n relatively prime to it. In particular, $a^{p-1} \equiv 1 \bmod p$ if p is prime. (The smallest k such that $a^k \equiv 1 \bmod n$ is called the order $ord_n(a)$ of a mod n, and obviously $ord_n(a)|\phi(n)$.)

- **Wilson's Theorem:** $(n-1)! \equiv 1 \bmod n$ iff n is prime. For composite n other than 4, $n|(n-1)!$.
- **Dirichlet's Theorem:** If $(a,b) = 1$, there are infinitely many primes p such that $p \equiv a \bmod b$.

That should be enough for these problems:

1. (MOP 2008 test) Let p and q be distinct prime numbers. Given that q divides $1 + n + \cdots + n^{p-1}$ for some integer n, prove that $q \equiv 1 \bmod p$.

2. (MOP 2008 test) Show that there exists a 17-term nonconstant arithmetic progression of positive integers such that their product is a perfect 2008th power.

Diophantine equations also pop up often. Solving these won't require any techniques you don't know, but it most commonly involves approaches such as considering small cases or special cases (e.g. relatively prime integers), reducing modulo a convenient n, and relating to a known equation such as $a^2 + b^2 = c^2$. Examples:

3. (MOP 2008 Team Contest) Prove that there are infinitely many pairs of distinct positive integers x, y such that $(x^3 + y^2)|(x^2 + y^3)$.

4. (UK 1998) Let x, y, z be integers such that $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$ and let $d = (x, y, z)$. Prove that $dxyz$ and $d(y - x)$ are perfect squares.

Functional Equations

A perennial favorite topic of olympiad problem writers, who place disproportionate emphasis upon it. Most functional equation problems will ask for an explicit function or class of functions satisfying an equation over $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. The best approaches then are

- Set variables equal to 0 or 1; these are usually the easiest function values to find.
- Equate different variables in the equation, so as to get cancellation.
- Once you've hypothesized a functional expression for the integers, try proving it by induction.
- If you've found an expression for $f(qx)$ in terms of $f(x)$, set $1 = qx$ to extend to all rationals $\frac{p}{q}$.

Most functional equations have solutions that are self-evident after minimal inspection; the trick lies in proving rigorously that there are no other functions satisfying them. An example of such an equation is **Cauchy's functional equation** $f(a + b) = f(a) + f(b)$, whose only solutions over the rationals (and only continuous solutions over the reals) are $f = cx$ for constants c. This functional form can be derived easily by the approach above, and many other functional equations satisfied by $f = cx$ can be solved by reduction to Cauchy's:

1. (USAMO 2002) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - y^2) = xf(x) - yf(y)$ for all reals x, y.

However, the functional form isn't always so obvious, and in such cases you may have to *derive* the form using the approach above:

2. (MOP 2008) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying the equation

$$f(x+y) + f(x-y) = 2[f(x) + f(y) + 1].$$

And sometimes the self-evident solution isn't the only one:

3. (Mock Olympiad) Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(m + f(n)) = f(f(m)) + f(n)$.

Other Techniques and Topics

- For graph theory problems, be familiar with the basic terminology of **complete** and **connected** graphs, **trees** and **spanning trees**, **chromatic number**, and **Eulerian** and **Hamiltonian** paths (including every edge and every vertex respectively). Also know when and how to manipulate graphs by adding/deleting vertices and edges, changing coloring, etc. Problems:

  1. (USAMO 1989) 20 members of a tennis club are playing 14 two-person games among themselves, each member playing at least once. Prove that there must be a set of 6 games with 12 distinct players.

  2. (BAMO 2005) There are 1000 cities in the country of Euleria and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city traveling along these roads. Prove that the government of Euleria may pave some of the roads such that every city will have an odd number of paved roads leading from it.

- In solving algebra problems, there really aren't any algebraic manipulations you'll need that you don't already know. Just be sure to know Vieta's formulas relating the coefficients of a polynomial to its roots, and be familiar especially with the use of **roots of unity** in factoring polynomials - they're immensely helpful, as below:

  (MOP 2008) Compute the remainder when $x^{2008} + 2x^{1004} + 3x^{502} + 4x^{251} + 5$ is divided by $x^4 + x^3 + x^2 + x + 1$.

- Whenever you are asked to prove a statement true for all natural numbers, all whole numbers, or in general all integers greater than some n, you should consider **induction** as a solution technique. Sometimes the induction step is relatively simple and intuitive, as in

  1. (USAMO 2002) Let S be a set with 2002 elements and let N be an integer between 0 and 2^{2002} inclusive. Prove that it is possible to color every subset of S blue or red such that (1) the union of any two red subsets is red, (2) the union of any two blue subsets is blue, and (c) there are exactly N red subsets.

  But sometimes the inductive step is much more involved and it is less obvious that induction may even be a viable approach:

  2. (MOP 2008 test) Let P_n be a degree-n polynomial with real coefficients and let $t \geq 3$ be a real number. For integers $k \geq 0$, show that

  $$max_{0 \leq k \leq n+1}\{|t^k - P_n(k)|\} \geq 1.$$

- For some counting problems, the concept of **bijections** or one-to-one mappings is useful for equating two quantities:

  (St. Petersburg 1989) Tram tickets have six-digit numbers from 000000 to 999999. A ticket is called *lucky* if its first three digits have the same sum as its last three digits, and *medium* if the sum of all its digits is 27. Prove that the number of lucky tickets is equal to the number of medium tickets.

  While we're on the subject, it's useful to know that a bijective mapping is one that is both **injective** ($f(x) \neq f(y)$ implies $x \neq y$) and **surjective** (for every y in the codomain, there exists an x in the domain such that $f(x) = y$). Injectivity and surjectivity are also useful properties to establish in the solution of functional equations.

- Random unclassifiable problem lulz!

  (MOP 2008 Team Contest) Let a_i, b_i $(1 \leq i \leq n)$ be positive real numbers. Suppose $a_i < b_i$ for all i and

  $$b_1 + b_2 + \cdots + b_n < 1 + a_1 + \cdots + a_n.$$

  Prove that there exists a real number c such that for all i and all integers k, $(a_i + c + k)(b_i + c + k) > 0$.

Feel free to email me at the address at the top of page 1 regarding any problems we didn't go over today. Be aware that Gmail is currently failing on me, though.